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On the complexity of orthogonal compaction ☆

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Abstract

We consider three closely related optimization problems, arising from the graph drawing and the VLSI research areas, and conjectured to be NP-hard, and we prove that, in fact, they are NP-complete. Starting from an orthogonal representation of a graph, i.e., a description of the shape of the edges that does not specify segment lengths or vertex positions, the three problems consist of providing an orthogonal grid drawing of it, while minimizing the area, the total edge length, or the maximum edge length, respectively.

This result confirms a long surviving conjecture of NP-hardness, justifies the research about applying sophisticated, yet possibly time consuming, techniques to obtain optimally compacted orthogonal grid drawings, and discourages the quest for an optimally compacting polynomial-time algorithm.

Keywords: orthogonal compaction; orthogonal drawing; orthogonal representation; NP-complete problem; SAT reduction

1. Introduction

The orthogonal drawing standard is recognized to be suitable for several types of diagrams, including data flow diagrams, entity-relationship diagrams, state-transition charts, circuit schematics, and many others. Such diagrams are extensively used in real-life applications spanning from software engineering, to databases, real-time systems and VLSI.

A well known approach to produce orthogonal drawings is the topology-shape-metric approach (see, for example, [1,3,5,10,14]), in which the graph drawing process is organized in three steps (see Fig. 1).

Planarization step: determines the topology of the drawing, which is described by a planar embedding, i.e., the order of the edges around each vertex. The purpose of this step is to minimize edge crossings. Also, each crossing is replaced by a dummy vertex, so that the final topology is planar.

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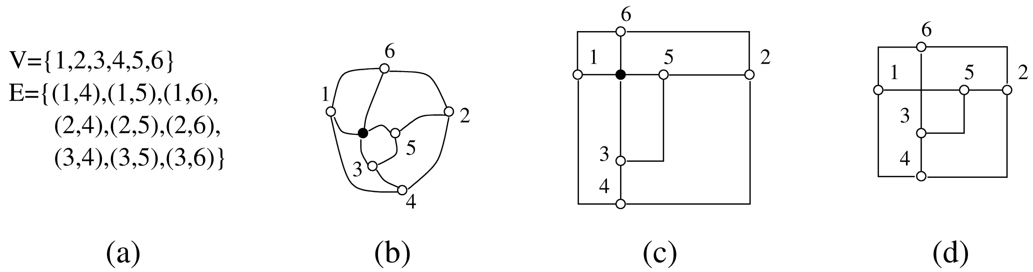


Fig. 1. The three steps of the topology-shape-metric approach: (a) abstract description of a graph, (b) planar embedding produced by the planarization step, (c) orthogonal representation produced by the orthogonalization step, and (d) final drawing produced by the compaction step. The dummy vertex (black) is introduced by the planarization step and removed at the end of the compaction step.

Orthogonalization step: determines the shape of the drawing, in which vertices do not have coordinates and each edge is equipped with a list of angles, describing the bends featured by the orthogonal line representing the edge in the final drawing. The purpose of this step is the reduction of the total number of bends.

Compaction step: determines the final coordinates of the vertices and bends. Also, at the end of this step the dummy vertices introduced in the planarization step are removed.

The name of the last step originates from the fact that during this step an aesthetic measure between area, total edge length, or maximum edge length is hopefully minimized. The compaction problem is precisely the optimization problem consisting of minimizing one of the three mentioned measures, while performing the compaction step: in particular we call Orthogonal Area Compaction (OAC), Orthogonal Total Edge Length Compaction (OTELC), and Orthogonal Maximum Edge Length Compaction (OMELC) the three problems, respectively.

Finding the intrinsic computational complexity of the compaction problem has been for a long time an elusive goal. Decades of intense research in the field of orthogonal graph drawing have not affected our knowledge in this respect: the problem is mentioned as open in recent papers as in foundational ones [8,11,15]. As far as we know, the only contribution to this subject is the early result contained in [4], where the trivial case of not connected graphs is demonstrated to be NP-complete.

The compaction problem has been one of the challenging tasks in the VLSI research field too, where the requirement of minimizing the size of a circuit layout while preserving its shape, led to formulations similar to those arising in the graph drawing area, although, for VLSI purposes, vertices are possibly replaced by squares and additional constraints (e.g., on the length of specific edges) are generally managed. Since several VLSI formulations, related with the compaction problem, are proved to be NP-hard [12], compacting orthogonal representations is widely believed to be an NP-hard problem too, and heuristics producing suboptimal solutions are usually applied in practical cases.

A first strain of heuristics descend from the "rectangular refinement" approach proposed in [14], based on the fact that the compaction problem is tractable when all faces of the orthogonal representation are rectangular, and consisting of splitting the non-rectangular faces into rectangles and removing the introduced edges after compaction. This approach may yield a linear time compaction step that minimizes the area, or an $O(n^{7/4}\log n)$ compaction step that minimizes the area and (secondarily) the total edge length [3].

Recently, the compaction step has been the subject of a renewed research interest. The problem of optimal compacting with respect to total edge length was approached with an ILP formulation in [9], relying on branch-and-cut or branch-and-bound techniques to find an optimal solution. Lately, a novel compaction method has been devised that optimizes with respect to the area (and, secondarily, total edge length) in polynomial time in the particular case of turn-regular orthogonal representations [2]. The latter approach gives rise to new heuristics based on a "turn regularization" rather than a "rectangularization" preprocessing step.

In this paper, by means of a reduction from the SAT problem, we prove that compacting an orthogonal representation of a connected graph, while minimizing an aesthetic measure between area, total edge length, and maximum edge length is an NP-complete problem. To accomplish this, after formally defining the three problems in Section 2, we introduce in Section 3 a class of orthogonal representations, that we call *sliding-rectangles gadgets*, admitting an exponential number of orthogonal grid drawings with minimum area, in all of which the basic blocks composing the gadget necessarily inherit the property of being themselves drawn with the minimum area. This property is exploited to build a sliding-rectangles gadget corresponding to a formula ϕ of the SAT problem. We will prove the NP-hardness of the OAC problem by showing that such orthogonal representation admits exclusively the subset the orthogonal grid drawings with the minimum area corresponding to the truth assignments satisfying the formula ϕ.

In Section 4 the result of Section 3 is extended to the problems of compacting an orthogonal representation while minimizing the total edge length and the maximum edge length. In Section 5 the three problems are proved to be in NP. Finally, Section 6 contains our conclusions.

2. Preliminaries

We assume familiarity with basic graph theoretic concepts and graph drawing terminology (see, e.g., [7] and [3], respectively) concerning planarity, planar graphs and planar representations.

In a *planar orthogonal drawing* Γ of a graph G, vertices are placed on distinct points of the plane and edges are alternating sequences of horizontal and vertical non-intersecting segments.

In what follows we will consider, without loss of generality, only orthogonal drawings with no bends, since each bend can be replaced by a dummy vertex of degree two. According to this assumption, our definition of orthogonal representation, although similar to the usual one (see, e.g., [2,3,14]), will be specialized for capturing the notion of "orthogonal shape" of a planar orthogonal graph without bends.

Let f be a face of a plane graph G of maximum degree four, and let Γ be an orthogonal drawing of G. We associate with each pair of consecutive edges (possibly coinciding) of f, a value α, such that $\alpha \cdot \pi/2$ is the angle formed by the two consecutive edges into f.

An orthogonal representation H of G describes an equivalence class of planar orthogonal drawings of G with "similar shape", that is, with the same α values associated with the angles around vertices of G. More formally, an *orthogonal representation* of a graph G is an assignment of integer values α to each angle formed by a pair of consecutive edges around each vertex of G satisfying the following properties:

(1) $1 \leqslant \alpha \leqslant 4$,
(2) for each vertex v, the sum of α values of its angles is 4,
(3) let $a(f)$ be the number of vertices incident to a face f; the sum of α values around face f is $2a(f) - 4$, or $2a(f) + 4$ depending whether f is an internal face or the external face.

An *orthogonal grid drawing* Γ of a graph G (without bends) is an orthogonal drawing such that vertex coordinates have integer values.

The *length* (*height*) of an orthogonal grid drawing Γ is the maximum difference between the x (y) coordinates of its vertices.

An (l,h)*-compactable orthogonal representation* is an orthogonal representation such that l is the minimum length of all its orthogonal grid drawings, and h is the minimum height between all orthogonal grid drawings with length l.

The *area* of an orthogonal grid drawing Γ is the product of its length and height. The *total edge length* of an orthogonal grid drawing is the sum of the lengths of its edges. The *maximum edge length* of an orthogonal grid drawing Γ is the maximum value of all its edge lengths.

This paper is concerned with the complexity of producing an orthogonal grid drawing Γ starting from its orthogonal representation H while minimizing the area of the drawing, the total edge length, or the maximum edge length. The three minimization criteria are considered to have roughly the same aesthetic effect: that of reducing the size of the drawing (or of part of it) and so improving its readability. However, conflicts between the three requirements (see Fig. 2) imply that they constitute three different, although closely related, optimization problems.

Following a standard technique (see, e.g., [6,13]), rather than address directly the three optimization problems we will consider their corresponding decision versions according to which the Orthogonal Area Compaction (Edge Length Compaction, Maximum Edge Length Compaction, respectively) problem consists in taking as input an orthogonal representation H of a graph G and a constant K, and deciding whether integer coordinates can be assigned to the vertices of G so that the area (total edge length, maximum edge length, respectively) of the drawing is less or equal than K. More formally they can be defined as follows.

Problem. **Orthogonal Area Compaction (OAC)**
Instance. An orthogonal representation H of a graph G and a constant K.
Question. Can integer coordinates be assigned to the vertices of G so that the **area** of the drawing is less or equal than the value of the constant K?

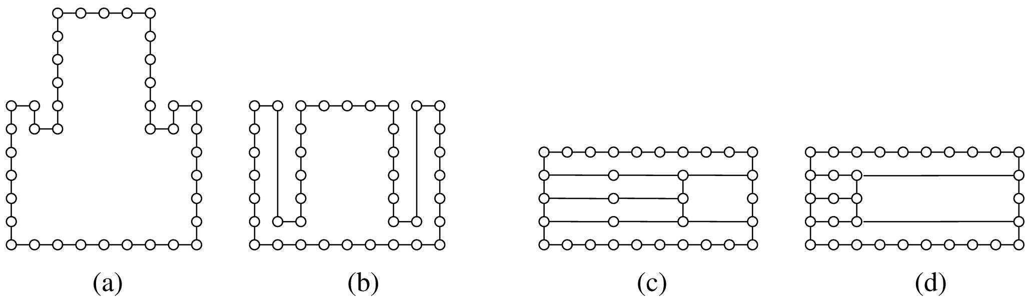


Fig. 2. The orthogonal drawings (a) and (b) correspond to the same orthogonal representation, and show how the two requirements of minimizing the area and minimizing the total (or maximum) edge length may not be met by a single drawing (the graph is biconnected and its orthogonal representation is “turn-regular” as defined in [2]). The drawings (c) and (d) too correspond to a single orthogonal representation: (c) minimizes the maximum edge length and (d) the total edge length (the graph is biconnected, its orthogonal representation is “turn-regular” and rectangular).

Problem. **Orthogonal Total Edge Length Compaction (OTELC)**
Instance. An orthogonal representation H of a graph G and a constant K.
Question. Can integer coordinates be assigned to the vertices of G so that the **total edge length** of the drawing is less or equal than the value of the constant K?

Problem. **Orthogonal Maximum Edge Length Compaction (OMELC)**
Instance. An orthogonal representation H of a graph G and a constant K.
Question. Can integer coordinates be assigned to the vertices of G so that the **maximum edge length** of the drawing is less or equal than the value of the constant K?

We will show in the following sections that the three problems above are NP-hard and are in NP. This is summarized in the following theorem.

Theorem 1. *The OAC, OTELC and OMELC problems are NP-complete.*

3. NP-hardness of the OAC problem

We prove that the Orthogonal Area Compaction problem is NP-hard by means of a reduction from the SAT problem:

Problem. **Satisfiability (SAT)**
Instance. A set of clauses, each containing literals from a set of Boolean variables.
Question. Can truth values be assigned to the variables so that each clause contains at least one true literal?

Given a formula ϕ in conjunctive normal form with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, we will produce an orthogonal representation $H_A(\phi)$ and a constant $K_A(\phi)$ such that an orthogonal grid drawing of area less or equal than $K_A(\phi)$ exists if and only if ϕ is satisfiable.

Notice that in the SAT definition all the variables in the same clause can be assumed to be different, since multiple occurrences can be trivially eliminated.

In what follows, by using the gadget introduced in Section 3.1, we will show how to build the instance (H_A, K_A) of the OAC problem corresponding to an instance ϕ of the SAT problem (Section 3.2), and prove that a solution to the OAC problem on instance (H_A, K_A) exists if and only if the corresponding instance ϕ of the SAT problem is satisfiable (Section 3.3).

3.1. Sliding-rectangles gadget

The main problem in the construction of an instance (H_A, K_A) is the fact that the property of being drawn with minimum area is a global property, regarding the whole drawing, and does not necessarily reflect on parts of it. Fig. 3(a) provides an example in which the area covered by the external box is minimum, while the subgraphs contained inside are not themselves drawn as small as they could be. This is a drawback since we would like to devise a chain of causes and effects leading from a minimized drawing to a satisfied formula and vice versa, but our only property seems to have no consequences if not on the boundary.

Obviously, we could change the size of the external box so to force the global optimality to imply a local optimality, as in Fig. 3(b), or add a suitable number of objects to the drawing to obtain the same effect (Fig. 3(c)), but by doing this we limit the number of optimal solutions, i.e., we tend to produce orthogonal representations admitting only one orthogonal drawing with the minimum area.

Instead, since we expect the compacted drawing to correspond to a satisfying assignment and vice versa, we are seeking for an orthogonal representation that admits as many compacted drawings as there are assignments that satisfy the formula (possibly an exponential number).

What we need is a systematic way to assure that the area optimality is inherited by the internal parts of the grid drawing while preserving a suitable degree of "freedom" (i.e., number of alternatives) for the orthogonal grid drawing of the whole graph.

To this purpose we produce the *sliding-rectangles gadget* of Fig. 4. We hypothesize that the inside subgraphs can be modeled by n contiguous $(3,h)$-compactable rectangles. Each rectangle can slide vertically with respect to the following and preceding ones. The box around the graph has top and bottom side $4n+4$ vertices long and right side $h+8$ vertices long; between the rectangles and the external boundary we place a *belt* consisting of a path of $4+8n$ vertices. The first rectangle, the belt and the external box are linked together as shown in Fig. 4. Instead of giving the α values for each

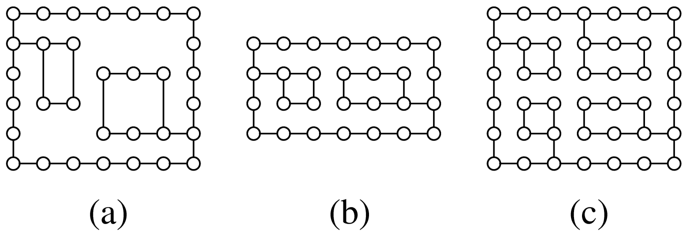


Fig. 3. While the orthogonal grid drawing in (a) occupies the minimum area, its subgraphs are not themselves drawn with minimum area. Conversely, in (b) and (c), the property of being drawn with minimum area is both a global and a local property.

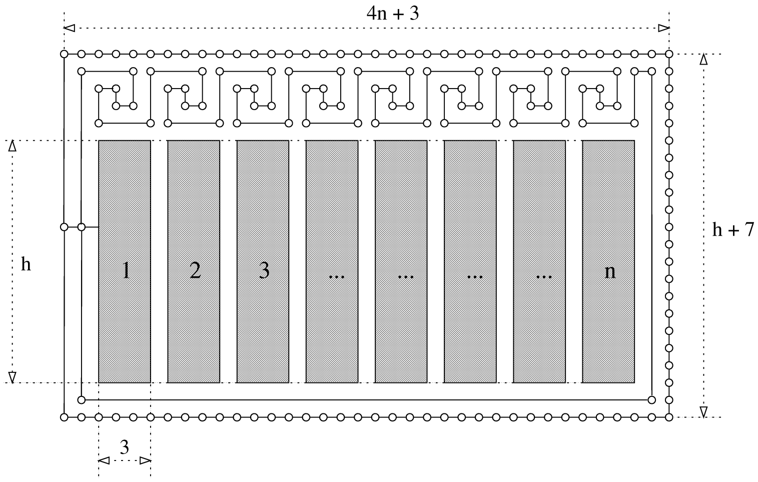


Fig. 4. A drawing in the minimum area of the sliding-rectangles gadget, in the particular case of $n=8$ and $h=14$.

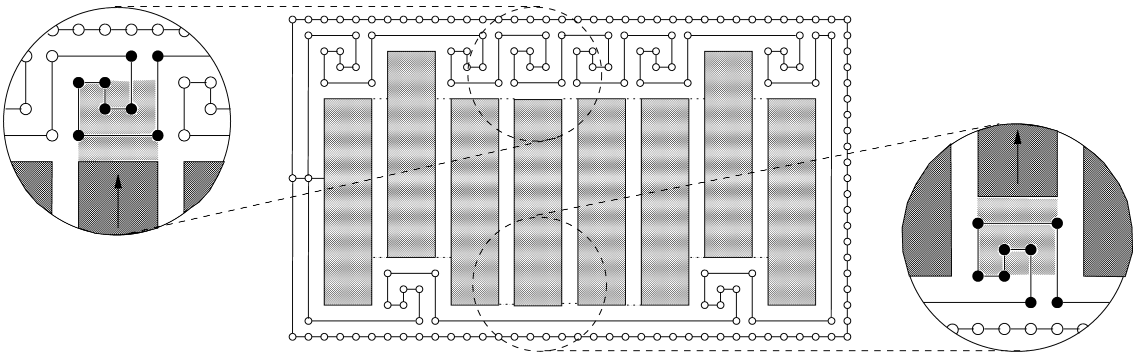


Fig. 5. (b) An orthogonal drawing with the minimum area of a sliding-rectangle gadget. When a rectangle slides from the "down" position to the "up" position, the darkened area in (a) is covered and the darkened area in (c) is available; a r^3l^4r subsequence (black vertices in (a)) can be removed, and a rl^4r^3 subsequence (black vertices in (c)), can be inserted in the turn sequence.

vertex of the belt, we will describe its angles specifying the *turn sequence* $\sigma = (r^4l^4)^n r^4$, where an r represents a right turn and an l represents a left turn, and r^n (l^n) represents a repetition of n right turns (left turns, respectively). The turn sequence σ succinctly describes the angles met when traversing the path clockwise starting from the vertex shared with the rest of the graph, and will be used in the following for the sake of brevity.

Note that, since four right turns are both at the beginning and at the end of the belt, the turn sequence $\sigma = (r^4l^4)^n r^4$, may be equivalently written as $r(r^3l^4r)^n r^3$ or $r^3(rl^4r^3)^n r$. It follows that removing a r^3l^4r subsequence has the same effect of removing a rl^4r^3 one. What's more, removing a r^3l^4r subsequence and inserting a rl^4r^3 one in the correct place (before the trailing r, for example) leaves the whole turn sequence unchanged.

The reason why we are interested in removing one subsequence and inserting the other is apparent from Fig. 5: suppose to slide a rectangle upwards of three horizontal grid lines, a r^3l^4r subsequence on the upper side may be removed, since it is partially covered by the rectangle. At the same time the new room made on the bottom side of the rectangle is exactly what is needed to host a rl^4r^3 subsequence.

Lemma 1. *In each orthogonal grid drawing of the sliding-rectangles gadget with the minimum area* $(4n+3) \times (h+6)$:
(1) *each rectangle is drawn in the minimum area, and*
(2) *each rectangle assumes necessarily one of the two positions "up" and "down" depicted in Fig.* 5.

Proof. Consider the areas available to the belt above and below each rectangle in an orthogonal grid drawing with the minimum area of the sliding-rectangles gadget. If the rectangle is drawn itself in the minimum area and is in the down (up) position, 8 vertices of the belt can be placed in the space above (below), as shown in Fig. 5(b).

To prove the statement we need to show that the belt needs all the areas above or below the rectangles, and that, if a rectangle has an intermediate position, the two areas above and below the rectangle cannot host the same number of vertices of the belt.

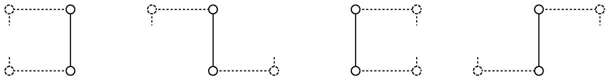


Fig. 6. The vertical segments of the belt are of the four type t_1, t_2, t_3 and t_4, represented in (a), (b), (c) and (d), respectively.

Observe that the edges of the belt are alternating horizontal and vertical segments, and the vertical edges alternate between the four types t_1, t_2, t_3 and t_4, depicted in Figs. 6(a), 6(b), 6(c) and 6(d), respectively.

Suppose that the rectangle is in the upper or lower position. Four horizontal grid lines are available to the belt. We prove that no more than eight vertices of the belt can be hosted in the four horizontal grid lines. In fact, from Fig. 6 it is apparent that a type t_1 or a type t_3 vertical edge needs at least three grid points of the vertical grid line it belongs to, and thus a vertical grid line hosting a type t_1 or a type t_3 vertical edge cannot host another vertical edge. The same holds for a vertical grid line hosting a type t_2 or t_4, since this would force the belt to traverse the vertical grid line two times in opposite directions, and therefore being trapped in the same side of the grid line, or to traverse it two times in the same direction, which is absurd. Since each vertical segment takes one vertical grid line, all the space above or below the rectangles is used by the belt.

If a rectangle has an intermediate position it leaves less than four horizontal grid lines to the belt in the space above and below the rectangle, and fails to host as many vertices (observe, for example, that a type t_1 or t_3 vertical edge cannot be hosted in such a space, since the belt would be trapped in the same side of the vertical grid line hosting the edge).

It follows that an orthogonal grid drawing in the minimum area of the sliding-rectangles gadget in which a rectangle is not drawn itself in the minimum area or has an intermediate position does not exist. □

From the above lemma, it follows that the sliding-rectangles gadget admits an exponential number of orthogonal grid drawings with the minimum area. The following two remarks show that this is not always necessarily the case.

Remark 1. For the sake of simplicity, in the gadget description given above the rectangles are supposed to be unconnected and independent. Since no part of the belt passes in between two rectangles, this assumption is unnecessary. The whole set of rectangles could be replaced by a suitable subgraph which behaves at the boundary *as if* composed by $(3, h)$-compactable sliding rectangles. This is important since in the following we will need to consider connected graphs (the problem of compacting unconnected graphs is trivially proven to be NP-hard by reducing the bin packing problem to it [4]).

Remark 2. In the hypothesis of Remark 1, the rectangles, once connected, may be allowed to assume only a subset of all the possible (otherwise exponential) "up" and "down" configurations. However, considerations analogous to those contained in the proof of Lemma 1 assure that no intermediate position can be assumed by the rectangles, and that they are themselves drawn in the minimum area. An accordingly modified version of Lemma 1 holds, with the additional statement that in the drawings with minimum area of the gadget only the allowed "up" and "down" configurations will be found.

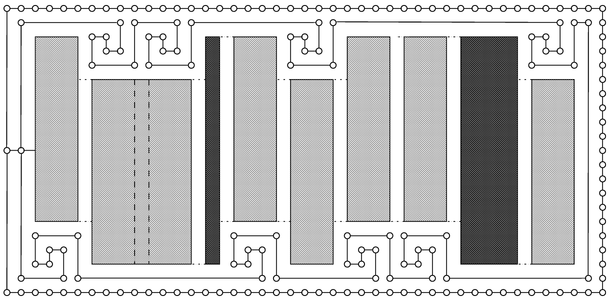


Fig. 7. A variant of the sliding-rectangles gadget obtained by inserting two immovable rectangles (Property 1) and replacing a $(3, h)$-compactable sliding rectangle with a $(7, h)$-compactable one (Property 2).

The following properties introduce some further variants to the sliding-rectangles gadget, for which an accordingly modified version of Lemma 1 holds.

Property 1. *In the sliding-rectangles gadget, a $(w', h + 3)$-compactable rectangle, with w' arbitrary, can be inserted at any position between the sliding-rectangles, provided that $w' + 1$ vertices are added to the top and bottom side of the external box.*

Proof. To prove the statement it suffices to observe that the drawings with minimum area of the modified gadget are in one-to-one correspondence with the drawings with minimum area of the original one. Namely, one drawing can be obtained from the other by removing or inserting the vertical stripe comprising the $(w', h+3)$-compactable extra rectangle and the additional vertices of the top and bottom side of the external box. By Lemma 1 in any drawing with minimum area the sliding rectangles are drawn in the minimum area, and are forced to assume necessarily one of the two positions "up" or "down". □

Property 2. *In the sliding-rectangles gadget, a $(3, h)$-compactable rectangle can be replaced by a $(3+4c, h)$-compactable one, where c is an arbitrary positive integer, provided that $4c$ vertices are added to the top and bottom side of the external box, and a $(r^4 l^4)^c$ subsequence is inserted at the beginning of the turn sequence of the belt.*

Proof. Given a sliding-rectangles gadget with n $(3, h)$-compactable rectangles, the operations described in the statement produce an orthogonal representation that is equivalent to a sliding-rectangles gadget with $n + c$ $(3, h)$-compactable rectangles, with the additional constraint that $c + 1$ adjacent rectangles are forced to be all in the same "up" or "down" position. By Lemma 1 and by Remark 2 in any drawing with the minimum area of the whole gadget the rectangles are forced to be drawn themselves in the minimum area and to not to be in an intermediate position. □

Fig. 7 shows a sliding-rectangles gadget featuring both variants.

3.2. Instance (H_A, K_A) construction rules

In this subsection we describe how to construct an instance $(H_A(\phi), K_A(\phi))$ of the OAC problem corresponding to an instance ϕ of the SAT problem, in such a way as to incorporate a sliding-rectangles gadget.

The construction of the orthogonal representation $H_A(\phi)$ requires three steps:

(i) build a clause-gadget for each clause C_i,

(ii) combine clause-gadgets together, and

(iii) add external boundary and belt.

These three steps are described in the following three paragraphs. The fourth paragraph is concerned with producing a value for $K_A(\phi)$.

(i) Clause-gadget construction

In the following we assume that the formula ϕ of the SAT problem has n Boolean variables, $x_1, \ldots, x_n$, and m clauses $C_1, \ldots, C_m$.

The clause-gadget is composed by n *chambers*, one for each variable, whether the variable actually occurs in the clause C_i or not. We call (i, j)-chamber the chamber of clause C_i corresponding to the variable x_j. The (i, j)-chamber, with $1 < j < n$ is shown in Fig. 8(a), while the $(i, 1)$-chamber and the (i, n)-chamber are shown in Figs. 8(b) and 8(c), respectively.

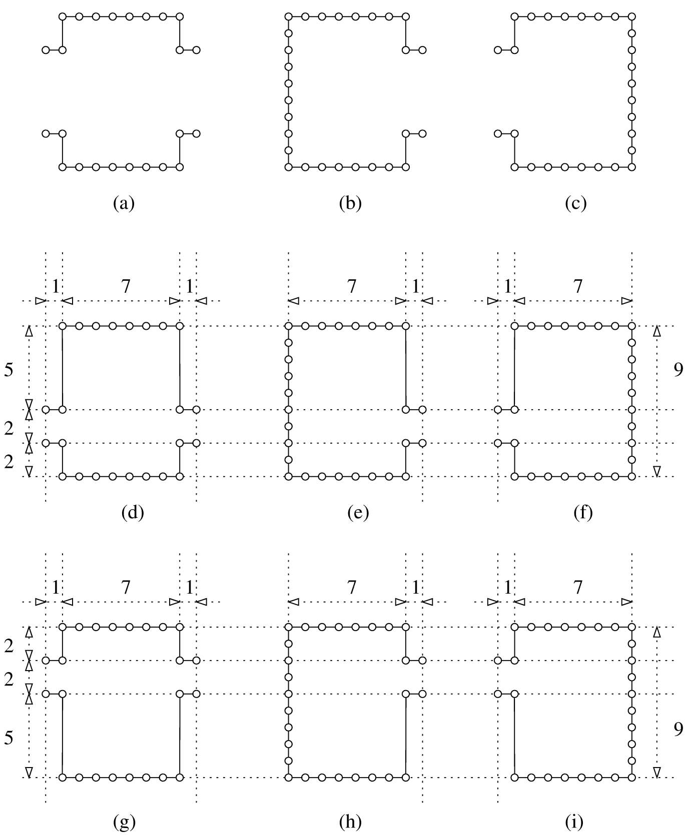


Fig. 8. First line: the chambers corresponding to (a) a variable x_j, with $1 < j < n$, (b) the variable x_1 and (c) the variable x_n. Second line: true-compliant orthogonal grid drawings corresponding to the orthogonal representations of figures (a), (b) and (c), respectively. Third line: false-compliant orthogonal grid drawings corresponding to the same orthogonal representations.

Observe that the edge lengths of Figs. 8(a)–(c) are not meaningful, since such figures are only meant to describe orthogonal representations. However, some peculiar orthogonal grid drawings of the chambers will be so recurrent in what follows to deserve a definition: we define *true-compliant* an orthogonal grid drawing of a chamber such that the vertex distances are exactly those represented in Figs. 8(d)–(f), and *false-compliant* an orthogonal grid drawing of a chamber such that the vertex distances are exactly those represented in Figs. 8(g)–(i).

For the sake of brevity we call *compliant* a true-compliant or false-compliant orthogonal grid drawing of a chamber, and we say that, in a given orthogonal grid drawing Γ of H_A, a chamber is true-compliant (false-compliant, compliant, respectively) whenever the orthogonal grid drawing of the chamber induced by Γ is true-compliant (false-compliant, compliant, respectively).

All the n chambers corresponding to clause C_i are attached together in a row, in such a way that the (i, j)-chamber shares two vertices with the $(i, j+1)$-chamber. We call such vertices *weldings*. Fig. 9 shows all the chambers of a clause-gadget for a formula with six variables.

To be completed the clause-gadget is added with two types of further subgraphs: obstacles and pathways.

An (i, j)-chamber corresponding to a variable x_j not occurring in the clause C_i receives an *obstacle* as shown in Fig. 9(b). Any other (i, j)-chamber receives two obstacles as shown in Fig. 9(c), if the variable x_j occurs with a positive literal, or as shown in Fig. 9(d), otherwise. Fig. 10(a) shows an example of a clause-gadget with its obstacles.

Finally, the clause-gadget is augmented with a *pathway*. The role of the pathway is that of forcing the clause-gadget to admit only certain orthogonal drawings with all the chambers drawn compliant. Namely, drawings in which all true literals correspond to false-compliant chambers and all false literals correspond to true-compliant chambers will not be admitted. The pathway is composed by a succession

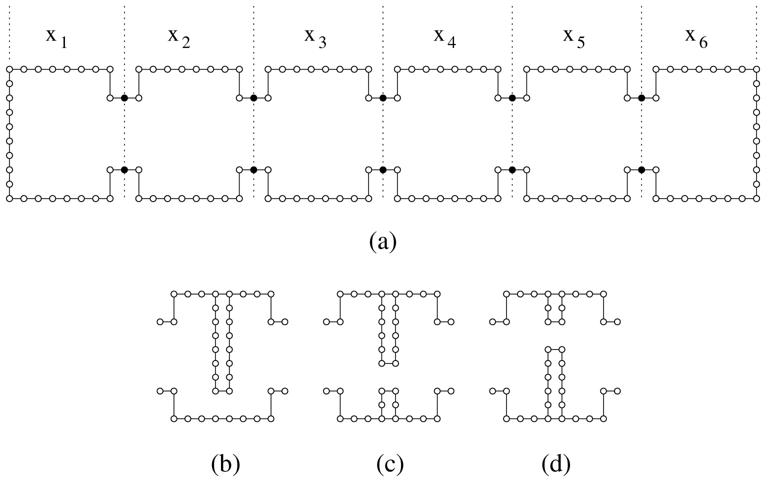


Fig. 9. (a) The chambers corresponding to a clause of a formula with six variables. The black vertices are weldings. The obstacles inserted in a (i, j)-chamber, when variable x_j does not occur in clause C_i (b), when variable x_j occurs in the clause C_i with a positive literal (c), or a negative literal (d). The figure shows only the case of an (i, j)-chamber with $1 < j < n$; other cases are similar.

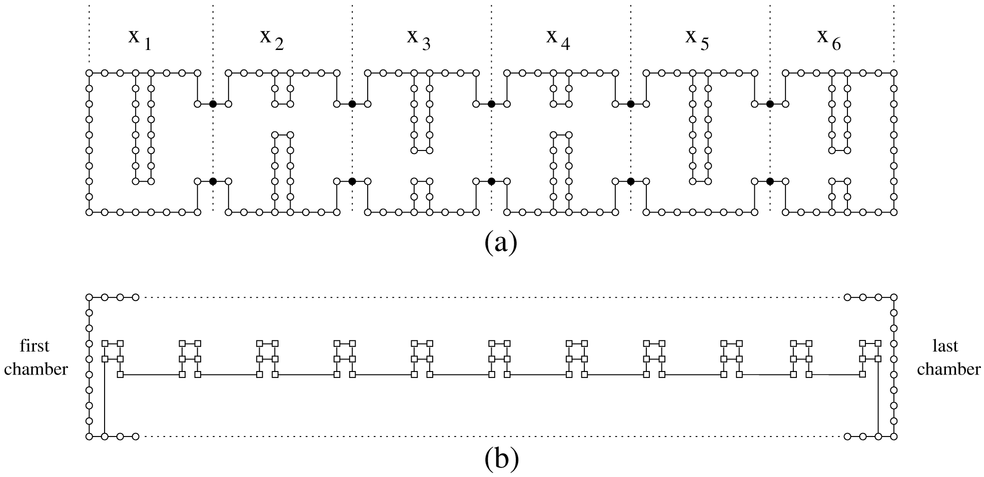


Fig. 10. (a) The chambers corresponding to clause $C_i = \overline{x}_2 \vee x_3 \vee \overline{x}_4 \vee x_6$ of a formula ϕ with six variables once completed with obstacles. The black vertices are weldings. (b) The pathway inserted in the clause-gadget is composed by 11 (i.e., $2n - 1$) A-shaped structures linked together.

of $2n-1$ A-shaped structures linked together as shown in Fig. 10(b). It originates from the $(i,1)$-chamber and terminates in the (i,n)-chamber, as shown in the same figure.

(ii) Combining clause-gadgets together

All the clause-gadgets corresponding to formula ϕ are placed one upon the other, so that each (i, j)-chamber shares its bottom 8 vertices with the $(i+1, j)$-chamber, for $i = 1, \ldots, n-1$. Furthermore, hinges are introduced. *Hinges* are vertical paths, originating from the weldings.

A hinge 8 vertices long links the welding between the (i, j)-chamber and the $(i, j+1)$-chamber with the welding between the $(i+1, j)$-chamber and the $(i+1, j+1)$-chamber, for $i = 1, \ldots, m-i$ and $j = 1, \ldots, n-1$.

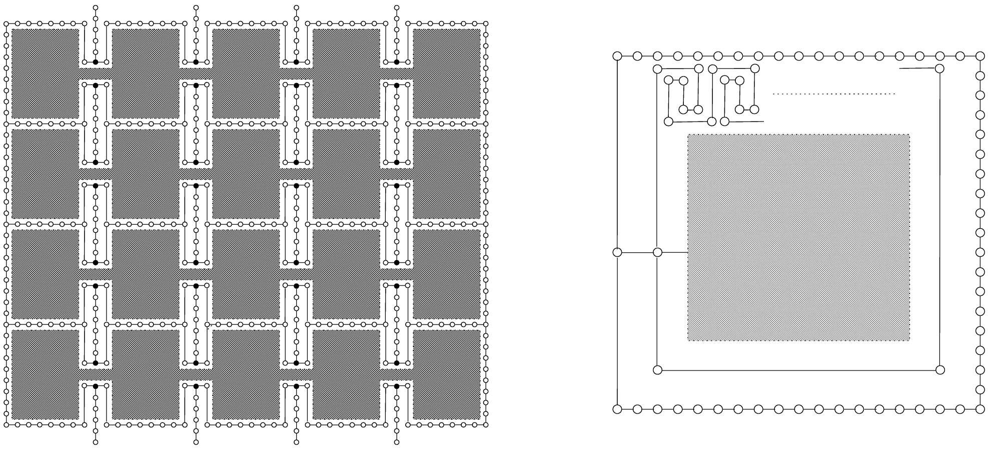
A hinge 6 vertices long attaches to the welding between the (i, j)-chamber and the $(i, j+1)$-chamber, with $i = 1$ or $i = n$, and $j = 1, \ldots, n-1$. The clause-gadgets and hinges for a formula with five variables and four clauses are shown in Fig. 11(a).

(iii) Adding external boundary and belt

To obtain the final orthogonal representation $H_A(\phi)$ an external boundary and a belt are added to the construction. The *external boundary* has a top and bottom side of $9n+3$ vertices and a right side of $9m+8$ vertices. The *belt* is a path inserted between the boundary and the core of the construction and composed by $2+24(n-1)$ vertices, so that its turn pattern is $(r^4l^4)^{2n}r^4$. The external boundary, the belt, and the core of the construction are attached together as shown in Fig. 11(b).

(iv) Computing constant $K_A(\phi)$

The instance (H_A, K_A) of the OAC problem is completely defined as the value of $K_A(\phi) = (9n+4) \times (9m+7)$ is assigned. Fig. 12 shows an example of $H_A(\phi)$ for a formula ϕ with four Boolean variables and four clauses with $K_A(\phi) = 40 \times 43 = 1{,}720$.



(a) (b)

Fig. 11. (a) Clause-gadgets and hinges for a formula with five variables and four clauses. The black vertices are weldings. The inside of the clause-gadgets is not represented (darkened areas). (b) Adding external boundary and belt to the construction of (a) (darkened area) to obtain the final orthogonal representation $H_A(\phi)$. Figures (a) and (b) are meant to represent orthogonal representations only: edge lengths are accidental.

3.3. Correctness

Here we prove that an orthogonal grid drawing Γ of area at most $K_A(\phi)$ can be found for the orthogonal representation $H_A(\phi)$ if and only if the corresponding instance ϕ of the SAT problem admits a solution.

The following properties hold.

Property 3. *An orthogonal grid drawing of the orthogonal representation $H_A(\phi)$ has length at least $9n + 4$, and height at least $9m + 7$.*

Proof. It suffices considering the number of vertices on the top side and right side of the external boundary of the orthogonal representation $H_A(\phi)$ (see Fig. 11(b)). $\square$

Property 4. *The horizontal (vertical) distance between a welding v and a vertex on one of the vertical (horizontal) sides of the external boundary of $H_A(\phi)$ is the same in every orthogonal grid drawing Γ of $H_A(\phi)$ with area $K_A(\phi)$.*

Proof. For the horizontal distance it suffices considering that a path p leading from a vertex on the left vertical side, to a vertex on the right vertical side of a clause gadget can be found in $H_A(\phi)$, such that (i) v belongs to p, (ii) the coordinates of the vertices of p are non-decreasing with respect to the x-axis, and (iii) the number of horizontal edges of p is $9n + 2$. Since the distance between the right vertical side of a clause gadget and the right vertical side of the whole drawing is at least two, and the horizontal edges have length at least one, the horizontal position of v in any drawing of length $9n + 4$ is fixed.

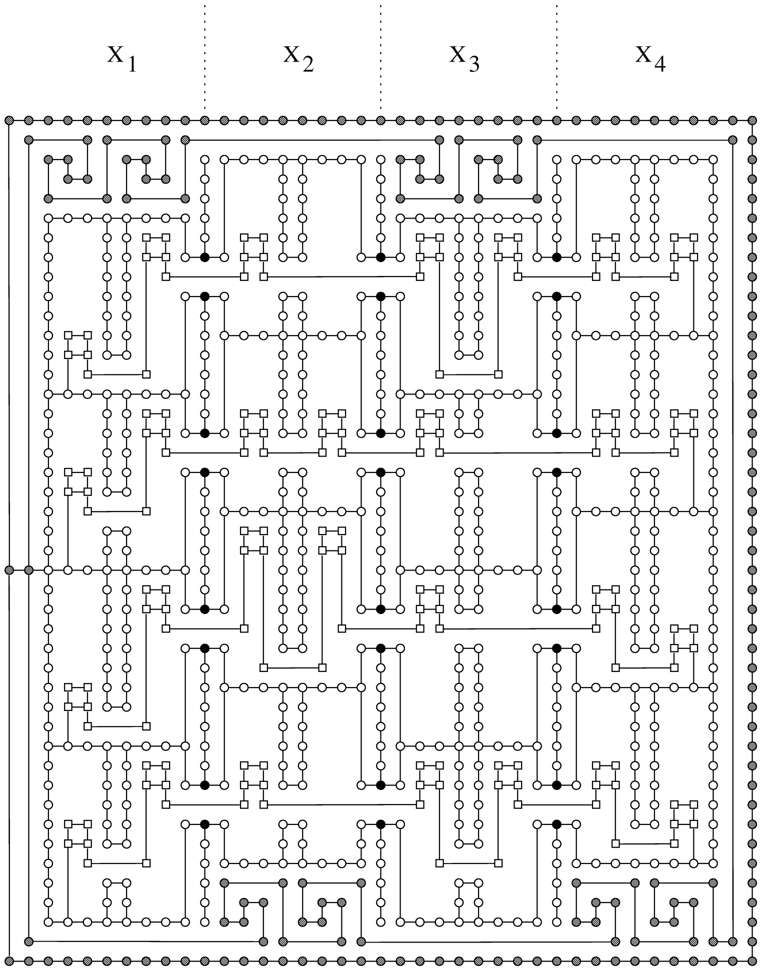


Fig. 12. The orthogonal representation $H_A(\phi)$ corresponding to the formula $\phi = (x_2 \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x}_3 \vee x_4) \wedge (\overline{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$. The peculiar orthogonal grid drawing shown in the figure has minimum area and corresponds to the truth assignment: $x_1 = \mathit{false}$, $x_2 = \mathit{true}$, $x_3 = \mathit{false}$, $x_4 = \mathit{true}$.

Regarding the vertical distance between a welding and the external boundary, notice that, since the area of the orthogonal grid drawing Γ is minimum, all the welding between (i, j)-chamber and $(i, j+1)$-chamber, with $i = 1, \ldots, n$ have the same x coordinate, i.e., the hinges that attach to them lie necessarily on the same vertical grid line. Considering the length of the hinges, and that at least a horizontal grid line must be left between two hinges, to host the pathway, and between the external boundary and the hinges, to host the belt, the statement follows. □

Property 5. *An orthogonal grid drawing of an* (i, j)*-chamber is* $(9, 9)$*-compactable.*

Proof. For the minimum width it suffices considering the number of vertices on the top side of the subgraphs represented in Figs. 8(a)–(c).

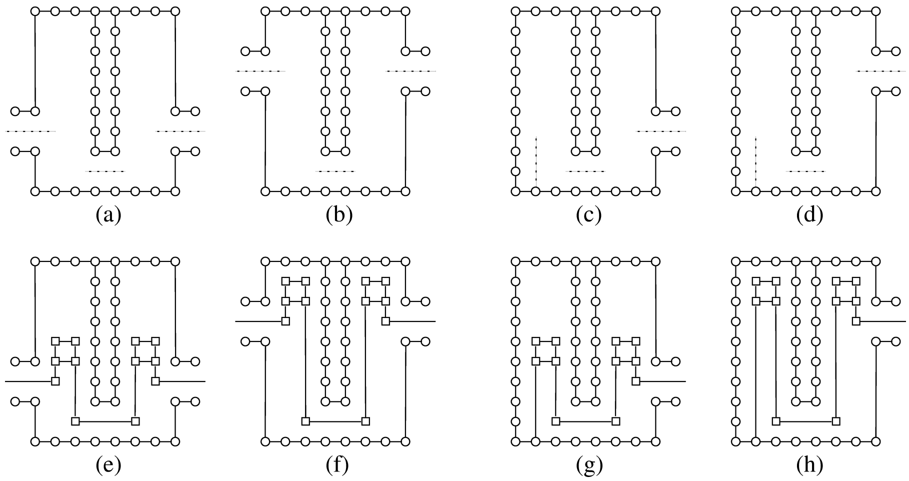


Fig. 13. The first line shows the possible configurations of a compliant (i, j)-chamber corresponding to a variable x_j not occurring in the clause C_i. The second line shows that the chambers necessarily contain two A-shaped structures of the pathway when are drawn in the minimum area.

To prove that when the chambers are drawn in the minimum width their minimum height is 9, for $j = 1$ or $j = n$ it suffices to consider the number of vertices on the left and right side of Figs. 8(b)–(c), respectively. For other values of j, observe that the chamber is composed by two non-connected subgraphs, and that at least a line must lay between them to host the pathway. □

Property 6. *A compliant (i, j)-chamber corresponding to a variable x_j not occurring in clause C_i contains two A-shaped structures of the pathway.*

Proof. The statement follows from the observation that the pathway must necessarily overlap with the dotted lines shown in Fig. 13. From the same figure is apparent that the only way to accomplish this is by inserting two A-shaped structures of the pathway inside an (i, j)-chamber. □

Property 7. *A true-compliant (false-compliant) (i, j)-chamber contains two A-shaped structures of the pathway if the corresponding literal is negative (positive), and may contain only one if the literal is positive (negative).*

Proof. The proof is obvious considering Figs. 14 and 15 where the cases are represented. □

Let Γ be an orthogonal grid drawing of H_A. We say that a clause-gadget C_i is *compliant in Γ* if each chamber of C_i is compliant in Γ. Also, we define *truth configuration of C_i in Γ* as the succession of Boolean values b_j, $j = 1, \ldots, n$, such that b_j is true (false) if the corresponding (i, j)-chamber is true-compliant (false-compliant).

Lemma 2. *A clause-gadget admits a truth configuration T if and only if assigning the sequence of Boolean values of T to the variables $x_1, \ldots, x_n$ produces at least one true literal in the corresponding clause C_i.*

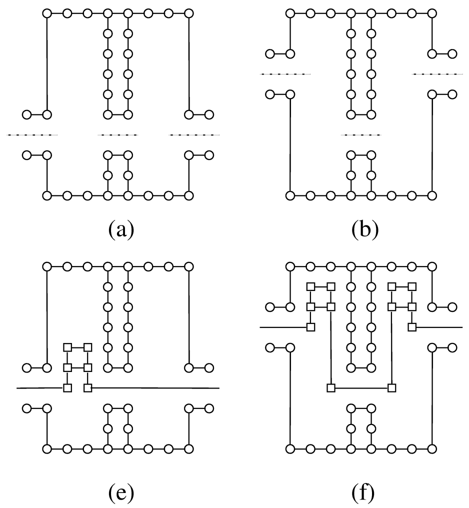


Fig. 14. When variable x_j occurs in the clause C_i with a positive literal, a true-compliant (i, j)-chamber ((a) and (c)), may contain only one A-shaped structure of the pathway ((e) and (g), respectively), while a false-compliant (i, j)-chamber ((b) and (d)) contains necessarily two A-shaped structures ((f) and (g), respectively).

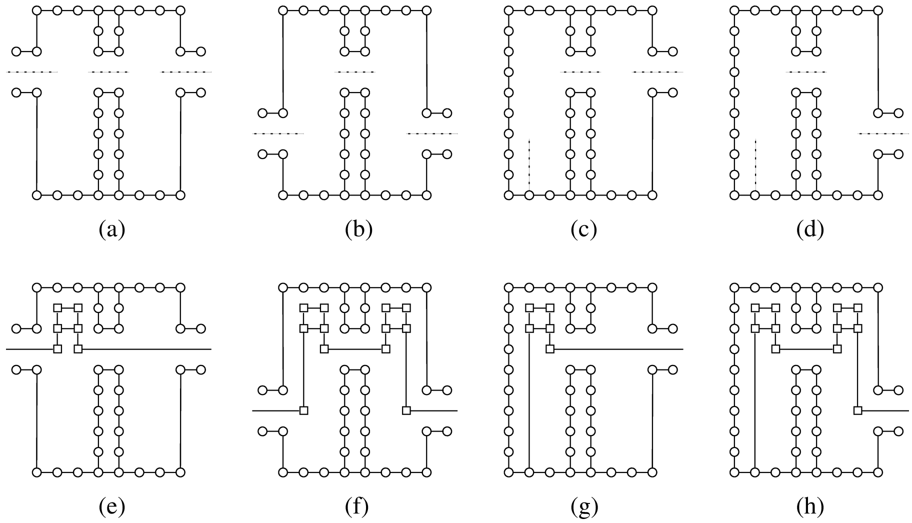


Fig. 15. When variable x_j occurs in the clause C_i with a negative literal, a false-compliant (i, j)-chamber ((a) and (c)), may contain only one A-shaped structure of the pathway ((e) and (g), respectively), while a true-compliant (i, j)-chamber ((b) and (d)) contains necessarily two A-shaped structures ((f) and (g), respectively).

Proof. By contradiction: suppose that the clause-gadget admits a truth configuration T, and that the sequence of Boolean values of T assigned to the variables $x_1, \ldots, x_n$ yield a false value for all the literals of the corresponding clause C_i. Each chambers of the clause-gadget corresponds to a variable not occurring in C_i, or occurring with an opposite truth value. Thus, each chamber must contain two

A-shaped structures of the pathway (Properties 6 and 7, respectively). It follows that the number of A-shaped structures of the pathway of the clause-gadget should be $2n$, while is $2n - 1$.

Conversely, suppose T is a truth configuration that, when assigned to the variables $x_1, \ldots, x_n$, produces at least a true literal in the clause C_i. Let x_t be a variable yielding a true literal. For Property 7 the (i, t)-chamber admits a compliant orthogonal grid drawing containing only one A-shaped structure of the pathway. Since $2n - 2$ A-shaped structures and $n - 1$ chambers are left, their drawing may be compliant with the corresponding truth value of the truth configuration T, since two A-shaped structures of the pathway are contained in each of them (Properties 6 and 7). □

Lemma 3. *The orthogonal representation $H_A(\phi)$ admits an orthogonal grid drawing of area at most $K_A(\phi)$ if and only if formula ϕ is satisfiable.*

Proof. Suppose formula ϕ is satisfiable and let T a truth configuration corresponding to an assignment satisfying ϕ. Lemma 2 states that each clause-gadget admits an orthogonal grid drawing compliant with T. If all the clause-gadgets are drawn compliant with the truth assignment T, each chamber is drawn in the minimum area, and the vertical column of chambers corresponding to the same Boolean variable covers a rectangular area of $7 \times 9m$, so is assimilable to a $(3 + 4c, 9m)$-compactable rectangle with $c = 1$ (Property 2). Between each pair of contiguous sliding rectangles a $(0, 9m + 4)$-rectangle is inserted, as allowed by Property 1. Furthermore, according to the above two variants, the top and bottom side of the external boundary are $9n + 3$ vertices long, the right side of the boundary is $9m + 8$ vertices long, and the belt has a bend pattern $\sigma = (r^4 l^4)^{2n} r^4$. It follows that the orthogonal representation $H_A(\phi)$ is a sliding-rectangles gadget, and Lemma 1 assures that an orthogonal grid drawing with area $K_A(\phi)$ exists for each truth configuration corresponding to a truth assignment satisfying ϕ.

Conversely, suppose formula ϕ is not satisfiable. Lemma 2 implies that there is not a truth configuration that can be assumed by all clause-gadgets. Since each chamber is attached to the chamber below with its bottom-side vertices, it follows that in any orthogonal grid drawing of $H_A(\phi)$, at least one chamber is not compliant. As a consequence one of the following holds:
(1) a chamber has height greater than 9,
(2) a clause-gadget has length greater than $9n$, or
(3) all clause-gadgets have length equal to $9n$, and a column of hinges has height greater than $9m + 4$.
Each of the above three statements implies that the orthogonal grid drawing Γ of $H_A(\phi)$ has an area greater than $K_A(\phi)$. In fact.

- Case 1 implies that at least one column of chambers has height greater than $9m$, and Lemma 1 rules out the existence of an orthogonal grid drawing of the sliding-rectangles gadget with area $K_A(\phi)$ in which a rectangle has an area greater than $7 \times 9m$.
- Case 2 implies that the width of the whole orthogonal grid drawing is greater than $9n + 4$, and from Property 3 and the definition of $K_A(\phi)$, the statement follows.
- Finally, case 3 implies analogously that the height of the orthogonal drawing is greater that $9m + 7$. □

Lemma 4. *The OAC problem is NP-hard.*

Proof. The statement follows from Lemma 3 and from the fact that the orthogonal representation $H_A(\phi)$ has $O(n \times m)$ vertices, and its construction (and the computation of $K_A(\phi)$) can be done in polynomial time. □

4. NP-hardness of the OTELC and OMELC problems

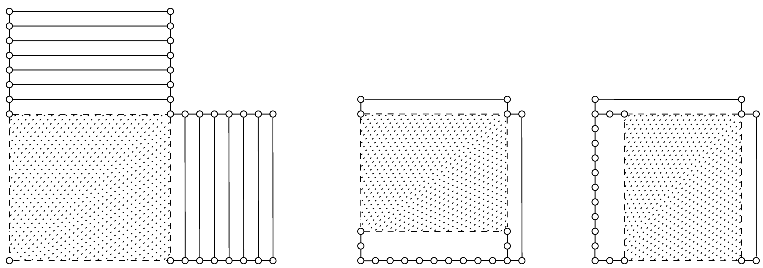
To prove that the Orthogonal Total Edge Length Compaction problem is NP-hard we reduce the SAT problem to it by slightly modifying the construction described in Section 3.

Observe that, in any orthogonal grid drawing of $H_A(\phi)$ with area $K_A(\phi)$, the total edge length cannot be greater than $\ell_0 = (l+1) \times (h+1)$, where l and h are the minimum values of the length and height of an orthogonal drawing of $H_A(\phi)$, respectively. To obtain the instance $(H_{\mathrm{TEL}}(\phi), K_{\mathrm{TEL}}(\phi))$ of the OTELC problem we add to $H_A(\phi)$ a number of ℓ_0 edges along the top and right sides of $H_A(\phi)$ and connect them to $H_A(\phi)$ as shown in Fig. 16(a). We assign to $K_{\mathrm{TEL}}(\phi)$ the value $\ell_0(l+2) + \ell_0(h+2) + \ell_0 = \ell_0(l+h+5)$.

If ϕ is satisfiable, then for Lemma 3 $H_A(\phi)$ admits an orthogonal grid drawing with area $K_A(\phi)$, and $H_{\mathrm{TEL}}(\phi)$ admits an orthogonal grid drawing with total edge length less or equal than $K_{\mathrm{TEL}}(\phi)$. Conversely, if $H_{\mathrm{TEL}}(\phi)$ admits an orthogonal grid drawing of total edge length less or equal than $K_{\mathrm{TEL}}(\phi)$, then $H_A(\phi)$ admits an orthogonal grid drawing with area $K_A(\phi)$. In fact, it's easy to see that every orthogonal grid drawing Γ of $H_{\mathrm{TEL}}(\phi)$ such that $H_A(\phi)$ covers an area bigger than $K_A(\phi)$ has a total edge length greater than $K_{\mathrm{TEL}}(\phi)$. Then, from Lemma 3 follows that the corresponding formula ϕ is satisfiable.

Similarly, the SAT problem can be reduced to the Orthogonal Maximum Edge Length Compaction problem. Namely, to obtain the instance $(H_{\mathrm{MEL}}(\phi), K_{\mathrm{MEL}}(\phi))$ of the OMELC problem we modify the orthogonal representation $H_A(\phi)$, adding a rectangular box to it, in such a way that the number of vertices along the top side of the obtained orthogonal representation is equal to the number of the vertices along the right side of it (see Figs. 16(b) and (c)). Finally, we add a pair of edges running along the top and right side of the construction as shown in the same figures.

Since the last added two edges are the longest in any orthogonal grid drawing of $H_{\mathrm{MEL}}(\phi)$, an orthogonal grid drawing of $H_{\mathrm{MEL}}(\phi)$ that minimizes the maximum length, also minimizes the perimeter of $H_A(\phi)$. In particular, when the maximum edge length of $H_{\mathrm{MEL}}(\phi)$ is $K_{\mathrm{MEL}}(\phi) = \max(9n+4, 9m+7)$, the perimeter of $H_A(\phi)$ is exactly $(9n+4) \times (9m+7)$, and the area of $H_A(\phi)$ is exactly $K_A(\phi)$, so that an orthogonal grid drawing of $H_{\mathrm{MEL}}(\phi)$ with maximum edge length equal to $K_{\mathrm{MEL}}(\phi)$ exists if and only if the corresponding formula ϕ is satisfiable.



(a) (b) (c)

Fig. 16. (a) The orthogonal representation $H_{\mathrm{TEL}}(\phi)$, and ((b) and (c)) the two possible cases for the orthogonal representation $H_{\mathrm{MEL}}(\phi)$. The darkened areas represent the $H_A(\phi)$ orthogonal representation of Fig. 11(b).

The following lemma is then proved.

Lemma 5. *The OTELC and OMELC problems are NP-hard.*

5. The OAC, OTELC and OMELC problems are in NP

To prove that the three problems are in NP we produce three nondeterministic Turing machines that decide them in polynomial time.

The nondeterministic Turing machines that we describe in the following take as input the instance (H, K), and generate the set S of orthogonal grid drawings of H with coordinates in the range $[0, v-1]$, where v is the number of vertices of H. Then they check each orthogonal grid drawing in S is to verify in polynomial time if its area, total edge length or maximum edge length, respectively, is less or equal than the constant K.

It's easy to show that, if an orthogonal grid drawing $\Gamma \notin S$ of the orthogonal representation H exists with area (total edge length, maximum edge length, respectively) less or equal than a constant K, then an orthogonal grid drawing $\Gamma' \in S$ exists with equal or less area (total edge length, maximum edge length, respectively). In fact, since our orthogonal representations have no bends, if Γ is an orthogonal grid drawing of H such that the horizontal (vertical) distance between two of its vertices is bigger than $v-1$, then Γ necessarily contains a vertical (horizontal) grid line non-intersecting any vertex of H that can be removed, decreasing the distance of such two vertices, the area, the total edge length and, possibly, the maximum edge length. Furthermore, if $\Gamma \notin S$ is an orthogonal grid drawing of H such that the distance of any two vertices is less or equal to $v-1$, then an orthogonal grid drawing $\Gamma' \in S$ exists with the same area (total edge length, maximum edge length, respectively).

Observe that, since the coordinates of the vertices are in the range $[0, v-1]$, it's easy to check in polynomial time whether an orthogonal grid drawing $\Gamma \in S$ is a feasible solution (vertices do not overlap, edges are orthogonal and do not intersect, angles around vertices are coherent with H labeling), and whether the area, total edge length or maximum edge length of Γ is less or equal than the constant K.

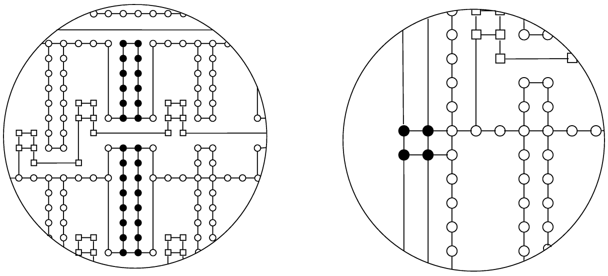
The nondeterministic Turing machine for the OAC problem works as follows: it takes as input the instance (H, K), and, if v is the number of vertices of H, writes an arbitrary sequence of v coordinate pairs in the range $[0, v-1]$. When this writing stops, the machine goes back and checks to see whether the string written is an orthogonal grid drawing, and, if so, whether its area is less or equal than K. Similar nondeterministic Turing machines can be easily devised for the OTELC and OMELC problems. The following lemma is, therefore, proved.

Lemma 6. *The OAC, OTELC and OMELC problems are in NP.*

6. Conclusions and open problems

In this paper we have shown that compacting an orthogonal representation while minimizing an aesthetic measure between area, maximum edge length and total edge length is an NP-complete problem.

An interesting topic is whether the three problems retain their complexity when focusing on peculiar classes of graphs. One may ask, for example, what is the influence of the connectivity properties of the



(a) (b)

Fig. 17. Hinges and belt attachment can be thickened as shown in (a) and (b), respectively, to make the orthogonal representation $H_A(\phi)$ biconnected.

graphs. For biconnected graphs, in spite of the fact that the proposed constructions are not biconnected (due to hinges and belt attachment), it is easy to modify these parts (thickening them as shown in Fig. 17) so to produce an orthogonal representation $H_A(\phi)$ of a biconnected graph.

Other interesting problems are the following: does an orthogonal representation, whose underlying graph is a simple cycle, retain the complexity of the three general problems? Does "turn-regularity" (defined in [2]) characterize the orthogonal representations for which the compaction problem is polynomial?

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